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MATH 1120 SPRING 2021
Conic Sections

1 Theorems

Theorem 1 (Rotation of Axes). Suppose the positive x and y axes are rotated counter-clockwise through an angle θ to produce the axes x' and y' respectively. Then the coordinates $P(x, y)$ and $P(x', y')$ are related by the following systems of equations.

$$\begin{cases} x = x'\cos(\theta) - y'\sin(\theta) \\ y = x'\sin(\theta) + y'\cos(\theta) \end{cases}$$

$$\begin{cases} x' = x\cos(\theta) + y\sin(\theta) \\ y' = -x\sin(\theta) + y\cos(\theta) \end{cases}$$

Theorem 2. The equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ with $B \neq 0$ can be transformed into an equation with variables x' and y' without any $x'y'$ terms by rotating the x and y axes counter-clockwise through an angle θ which satisfies:

$$cot(2\theta) = \frac{A - C}{B}$$

Theorem 3. Suppose the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ describes a non-degenerate conic section.

- If $B^2 - 4AC > 0$ then the graph of the equation is a hyperbola.
- If $B^2 - 4AC = 0$ then the graph of the equation is a parabola.
- If $B^2 - 4AC < 0$ then the graph of the equation is an ellipse or circle.

Note: A non-degenerate conic section refers to a graph that is either a circle, parabola, ellipse, or hyperbola.

2 Problems 1–16

In exercises 1–16, graph the following equations.

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1 $x^2 + 2xy + y^2 - x\sqrt{2} + y\sqrt{2} - 6 = 0$

Solution: Strap in because this is going to be a long one. First thing I did here was use **theorem 3** to ascertain the end result of the graph of this guy. Really this isn't important to do this early on, but it can't hurt to know where we're going.

To do this let's first break down our coefficients.

$$A = 1 \qquad B = 2 \qquad C = 1$$
$$D = -\sqrt{2} \qquad E = \sqrt{2} \qquad F = -6$$

From here we use the equation from **theorem 3** and see what our relationship to 0 is!

$$\begin{aligned} B^2 - 4 \cdot A \cdot C &= 0 \\ 2^2 - 4 \cdot 1 \cdot 1 &= 0 \\ 4 - 4 &= 0 \end{aligned}$$

From this we know at the end of this insanity we will have a parabola! If we don't we can safely say we went wrong somewhere. Now, to officially get started. Next, we're going to make use of **theorem 2** to figure out our value of θ. Why am I going backwards with theorems you may be wondering. It's because I have no idea what I'm doing.

$$\begin{aligned} cot(2\theta) &= \frac{A - C}{B} \\ cot(2\theta) &= \frac{1 - 1}{2} \\ cot(2\theta) &= 0 \\ cot(\theta) &= 0 \text{ when } \theta = \frac{\pi}{2} \\ 2\theta &= \frac{\pi}{2} \\ \theta &= \frac{\pi}{4} + \frac{\pi}{2}k \text{ for any integer k} \end{aligned}$$

We choose $\frac{\pi}{4}$ here because it's easy and because I'm following the book and scared of what's to come. From here we can now tackle **theorem 1**. We have an angle to work with so we can actually define our x and our y. Or, at the least, start to.

$$\begin{aligned} x &= x' \cdot \cos(\theta) - y' \sin(\theta) \\ x &= x' \cdot \cos\left(\frac{\pi}{4}\right) - y' \sin\left(\frac{\pi}{4}\right) \\ x &= \frac{x' \cdot \sqrt{2}}{2} - \frac{y' \cdot \sqrt{2}}{2} \end{aligned}$$

$$y = x' \cdot \sin(\theta) + y' \cos(\theta)$$
$$y = x' \cdot \sin\left(\frac{\pi}{4}\right) + y' \cos\left(\frac{\pi}{4}\right)$$
$$y = \frac{x' \cdot \sqrt{2}}{2} + \frac{y' \cdot \sqrt{2}}{2}$$

So, here's where things get fun. Before we proceed, for our own sanity we should go ahead and figure out in advance what x^2, xy, and y^2 are so we can plug those values instead of doing an absurd amount of work all at once.

Calculating x^2

$$x^2 = \left(\frac{x' \cdot \sqrt{2}}{2} - \frac{y' \cdot \sqrt{2}}{2}\right) \cdot \left(\frac{x' \cdot \sqrt{2}}{2} - \frac{y' \cdot \sqrt{2}}{2}\right)$$
$$x^2 = \frac{2 \cdot (x')^2}{4} - \frac{x'\sqrt{2} \cdot y'\sqrt{2}}{4} - \frac{x'\sqrt{2} \cdot y'\sqrt{2}}{4} + \frac{2 \cdot (y')^2}{4}$$

$$\boxed{x^2 = \frac{(x')^2}{2} - \frac{(2 \cdot (x'\sqrt{2})) \cdot (2 \cdot (y'\sqrt{2}))}{4} + \frac{(y')^2}{2}}$$

Calculating y^2

$$y^2 = \left(\frac{x' \cdot \sqrt{2}}{2} + \frac{y' \cdot \sqrt{2}}{2}\right) \cdot \left(\frac{x' \cdot \sqrt{2}}{2} + \frac{y' \cdot \sqrt{2}}{2}\right)$$
$$y^2 = \frac{2 \cdot (x')^2}{4} + \frac{x'\sqrt{2} \cdot y'\sqrt{2}}{4} + \frac{x'\sqrt{2} \cdot y'\sqrt{2}}{4} + \frac{2 \cdot (y')^2}{4}$$

$$\boxed{y^2 = \frac{(x')^2}{2} + \frac{(2 \cdot (x'\sqrt{2})) \cdot (2 \cdot (y'\sqrt{2}))}{4} + \frac{(y')^2}{2}}$$

Calculating xy

$$xy = \left(\frac{x' \cdot \sqrt{2}}{2} - \frac{y' \cdot \sqrt{2}}{2}\right) \cdot \left(\frac{x' \cdot \sqrt{2}}{2} + \frac{y' \cdot \sqrt{2}}{2}\right)$$
$$xy = \frac{2 \cdot (x')^2}{4} + \frac{x'\sqrt{2} \cdot y'\sqrt{2}}{4} - \frac{x'\sqrt{2} \cdot y'\sqrt{2}}{4} - \frac{2 \cdot (y')^2}{4}$$

$$\boxed{xy = \frac{(x')^2}{2} - \frac{(y')^2}{2}}$$

That was a lot, but now this is where we really get going. What we need to do now is plug these values back into our original equation and see what it all boils down to.

Substituting everything back into the original equation:

$$
\begin{aligned}
&0 = x^2 + 2xy + y^2 - x\sqrt{2} + y\sqrt{2} - 6 \\
&0 = \left(\frac{(x')^2}{2} - \frac{(2\cdot(x'\sqrt{2}))\cdot(2\cdot(y'\sqrt{2}))}{4} + \frac{(y')^2}{2}\right) + \left(2\cdot\left(\frac{(x')^2}{2} - \frac{(y')^2}{2}\right)\right) + \\
&\left(\frac{(x')^2}{2} + \frac{(2\cdot(x'\sqrt{2}))\cdot(2\cdot(y'\sqrt{2}))}{4} + \frac{(y')^2}{2}\right) - \left(\left(\frac{x'\cdot\sqrt{2}}{2} - \frac{y'\cdot\sqrt{2}}{2}\right)\cdot\sqrt{2}\right) + \\
&\left(\left(\frac{x'\cdot\sqrt{2}}{2} + \frac{y'\cdot\sqrt{2}}{2}\right)\cdot\sqrt{2}\right) - 6
\end{aligned}
$$

Okay, so that looks horrifying. I hate it. Thankfully it isn't quite as bad as it looks. I'll tell you this now, most of it will cancel. It's still quite an undertaking so how I'll do it is by simplifying it in pieces. I'll be taking bits that are similar in format, mushing them together and seeing what happens. Firstly we'll start with the two biggest chunks in the parentheses. I'll be referring to each chunk within parentheses as chunk 1, 2, 3, 4 and 5 for my own sanity. So the two big chunks will be chunks 1 and 3 respectively.

$$\text{Chunks 1 and 3} = \frac{(x')^2}{2} - \frac{(2\cdot(x'\sqrt{2}))\cdot(2\cdot(y'\sqrt{2}))}{4} + \frac{(y')^2}{2} + \frac{(x')^2}{2} + \frac{(2\cdot(x'\sqrt{2}))\cdot(2\cdot(y'\sqrt{2}))}{4} + \frac{(y')^2}{2}$$

The big boys cancel out, leaving just the smaller fractions.

$$= \frac{2(x')^2}{2} + \frac{2(y')^2}{2}$$

$$\boxed{\text{Chunks 1 and 3} = (x')^2 + (y')^2}$$

See, that's not too bad. Chunk 2 simplifies very nicely as well!

$$\text{Chunk 2} = \left(2\cdot\left(\frac{(x')^2}{2} - \frac{(y')^2}{2}\right)\right)$$

$$\boxed{\text{Chunk 2} = (x')^2 - (y')^2}$$

Now it's just chunks 4 and 5.

$$
\begin{aligned}
\text{Chunks 4 and 5} &= \left(\left(\frac{x'\cdot\sqrt{2}}{2} - \frac{y'\cdot\sqrt{2}}{2}\right)\cdot\sqrt{2}\right) + \left(\left(\frac{x'\cdot\sqrt{2}}{2} + \frac{y'\cdot\sqrt{2}}{2}\right)\cdot\sqrt{2}\right) \\
&= \frac{x'\cdot 2}{2} - \frac{y'\cdot 2}{2} + \frac{x'\cdot 2}{2} + \frac{y'\cdot 2}{2} \\
&= x' - y' + x' + y'
\end{aligned}
$$

$$\boxed{\text{Chunks 4 and 5} = 2x'}$$

Putting all the chunks together:

$$(x')^2 + (y')^2 + (x')^2 - (y')^2 + 2x' - 6 = 0$$

$$2(x')^2 + 2x' - 6 = 0$$

Okay, this is now a quadratic! That means we're on the right track as this will give us a parabola. From here we just need to do some basic algebra. Thank goodness. Unfortunately we have a quadratic that can't be factored, so we will have to use the quadratic formula here. Our goal here is to, of course, figure out our x′-intercepts. These are different from x-intercepts as I will showcase later.

$$\begin{aligned} x' &= \frac{-6 \pm \sqrt{b^2 - 4 \cdot a \cdot c}}{2 \cdot a} \\ &= \frac{-2 \pm \sqrt{2^2 - (4 \cdot 2 \cdot -6)}}{2 \cdot 2} \\ &= \frac{-2 \pm \sqrt{52}}{4} \\ &= \frac{-2 \pm 2\sqrt{13}}{4} \\ x' &= \frac{-1 \pm \sqrt{13}}{2} \end{aligned}$$

$y' = 0$ when:

$$\boxed{x' = \frac{-1 + \sqrt{13}}{2} \text{ and} \frac{-1 - \sqrt{13}}{2}}$$

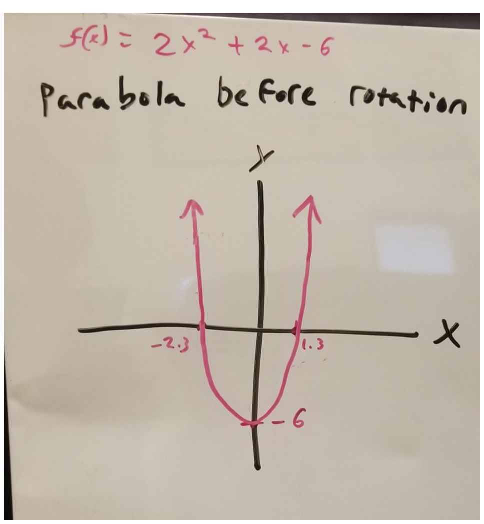
$$\boxed{x' \approx 1.3 \text{ and } -2.3}$$

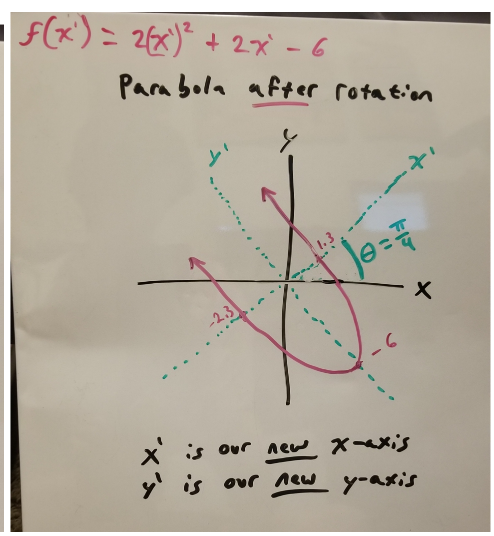
We can quickly solve for our y′-intercept as well. We just set $x = 0$

$$2(0)^2 + 2(0) - 6 = y'$$

$$y' = -6$$

From here we just plot our answer. You'll notice my answer differs from the one in the book. The books parabola is negative whereas mine is positive. I'll be upfront here, I have absolutely zero idea where I went wrong and this problem is far too long for me to go back and find out. This was my result.





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3 $5x^2 + 6xy + 5y^2 - 4\sqrt{2}x + 4\sqrt{2}y = 0$

Solution: First things first let's analyze our first three coefficients.

$$\boxed{A = 5} \boxed{B = 6} \boxed{C = 5}$$

$$B^2 - 4 \cdot A \cdot C$$
$$6^6 - (4 \cdot 5 \cdot 5)$$
$$36 - 100 < 0$$

Our answer being less than 0 implies that our answer will either be an ellipse or a circle. Next we need to ascertain how rotated our plot will be.

$$cot(2\theta) = \frac{A - C}{B}$$
$$cot(2\theta) = \frac{5 - 5}{6}$$
$$cot(2\theta) = 0$$
$$cot(\theta) = 0 \text{ when } \theta = \frac{\pi}{2}$$
$$2\theta = \frac{\pi}{2}$$
$$\theta = \frac{\pi}{4} + \frac{\pi}{2}k \text{ for any integer k}$$

How convenient, it's the same as the last problem. This actually has huge implications for the state of my sanity. This means we can skip calculating x, y, x^2, y^2 and xy because we're working with the same angle as last time. For the sake of clarity though let me just list the final results of those calculations below.

$$\boxed{x = \frac{x' \cdot \sqrt{2}}{2} - \frac{y' \cdot \sqrt{2}}{2}}$$

$$\boxed{y = \frac{x' \cdot \sqrt{2}}{2} + \frac{y' \cdot \sqrt{2}}{2}}$$

$$\boxed{x^2 = \frac{(x')^2}{2} - \frac{(2 \cdot (x'\sqrt{2})) \cdot (2 \cdot (y'\sqrt{2}))}{4} + \frac{(y')^2}{2}}$$

$$\boxed{y^2 = \frac{(x')^2}{2} + \frac{(2 \cdot (x'\sqrt{2})) \cdot (2 \cdot (y'\sqrt{2}))}{4} + \frac{(y')^2}{2}}$$

$$\boxed{xy = \frac{(x')^2}{2} - \frac{(y')^2}{2}}$$

From here we're back at the spicy bit, we need to place all of these back into our given equation. It'll be mostly the same as last time, but with different coefficients. We'll see how it works out!

$$0 = 5x^2 + 6xy + 5y^2 - 4\sqrt{2}x + 4\sqrt{2}y$$

$$0 = \left(5 \cdot \left(\frac{(x')^2}{2} - \frac{(2 \cdot (x'\sqrt{2})) \cdot (2 \cdot (y'\sqrt{2}))}{4} + \frac{(y')^2}{2}\right)\right) + \left(6 \cdot \left(\frac{(x')^2}{2} - \frac{(y')^2}{2}\right)\right) + \left(5 \cdot \left(\frac{(x')^2}{2} + \frac{(2 \cdot (x'\sqrt{2})) \cdot (2 \cdot (y'\sqrt{2}))}{4} + \frac{(y')^2}{2}\right)\right) - \left(\left(\frac{x' \cdot \sqrt{2}}{2} - \frac{y' \cdot \sqrt{2}}{2}\right) \cdot 4\sqrt{2}\right) + \left(\left(\frac{x' \cdot \sqrt{2}}{2} + \frac{y' \cdot \sqrt{2}}{2}\right) \cdot 4\sqrt{2}\right)$$

Alright, here we go. We'll break this into chunks again and simplify from there. First we'll start with the two largest chunks, 1 and 3.

Chunks 1 & 3:

$$\begin{aligned} \text{Chunks 1 and 3} = \frac{5(x')^2}{2} - \frac{(5 \cdot (x'\sqrt{2})) \cdot (5 \cdot (y'\sqrt{2}))}{4} + \frac{5(y')^2}{2} + \\ \frac{5(x')^2}{2} + \frac{(5 \cdot (x'\sqrt{2})) \cdot (5 \cdot (y'\sqrt{2}))}{4} + \frac{5(y')^2}{2} \\ = \frac{10(x')^2}{2} + \frac{10(y')^2}{2} \end{aligned}$$

$$\boxed{\text{Chunks 1 and 3} = 5(x')^2 + 5(y')^2}$$

Chunk 2:

$$\text{Chunk 2} = \left(6 \cdot \left(\frac{(x')^2}{2} - \frac{(y')^2}{2}\right)\right)$$

$$\boxed{\text{Chunk 2} = 3(x')^2 - 3(y')^2}$$

Chunks 4 & 5:

$$\begin{aligned} \text{Chunks 4 and 5} &= -\left(\left(\frac{x' \cdot \sqrt{2}}{2} - \frac{y' \cdot \sqrt{2}}{2}\right) \cdot 4\sqrt{2}\right) + \left(\left(\frac{x' \cdot \sqrt{2}}{2} + \frac{y' \cdot \sqrt{2}}{2}\right) \cdot 4\sqrt{2}\right) \\ &= -4x' + 4y' + 4x' + 4y' \end{aligned}$$

$$\boxed{\text{Chunks 4 and 5} = 8y'}$$

Chunks 1–5:

$$\begin{aligned} 5(x')^2 + 5(y')^2 + 3(x')^2 - 3(y')^2 + 8y' &= 0 \\ 8(x')^2 + 2(y')^2 + 8y' &= 0 \end{aligned}$$

Okay, so based on the work we did at the beginning we know we're working towards an ellipse or a circle. So, we need to tinker with our equation a bit to fit it into that format.

For a circle centered on (h, k), the standard equation for a circle is:

$$(x-h)^2 + (y-k)^2 = r^2$$

From this we can create equations for horizontal and vertical ellipses.

Horizontal:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Vertical:

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

Working our $8(x')^2$ into this format is trivial. This is the result.

$$\text{xchunk} = \frac{8(x'+0)^2}{1^2}$$
$$\text{xchunk} = \frac{(x'+0)^2}{\frac{1}{8}}$$

The ychunk is a little more exciting. We need to complete the square first for this guy.

$$2y^2 + 8y$$
$$2(y^2 + 4y)$$
$$2((y+2)^2) = 2(y^2 + 4y + 4)$$
$$2y^2 + 8y + 8$$

Now with some rearranging we get the following:

$$\frac{2(y'+2)^2}{1} - 8$$
$$\frac{(y'+2)^2}{\frac{1}{2}}$$

Now we smush the two together and using what we know about the anatomy of an ellipse we can sketch one out. We just need to add 1 to each side to get the format just right.

$$\frac{(x'+0)^2}{\frac{1}{8}} + \frac{(y'+2)^2}{\frac{1}{2}} - 7 = 1$$

Since the number under y′ is larger, we know we have a vertical ellipse. We also know it's centered 2 below the origin. As such we can sketch the ellipse below. Beforehand though let's solve for our vertices and co-vertices.

when $x = 0$:

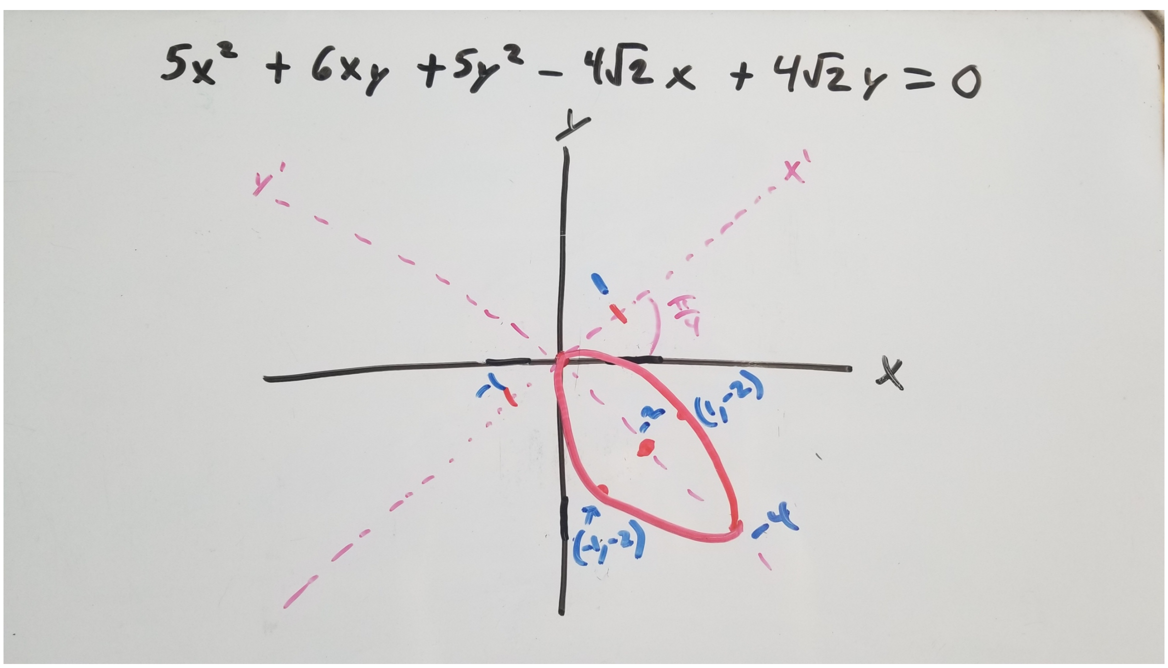
$$\frac{(y'+2)^2}{\frac{1}{2}} - 7 = 1$$
$$(y'+2)^2 = 4$$
$$y' + 2 = \pm 2$$

$$\boxed{y' = 0 \text{ or } -4}$$

when $y = -2$:

$$\frac{(x'+0)^2}{\frac{1}{8}} + \frac{(-2+2)^2}{\frac{1}{2}} = 8$$
$$(x')^2 + 0 = 1$$

$$\boxed{x' = \pm 1}$$



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Theorem 4. Suppose e and d are positive numbers. Then

- the graph $r = \frac{ed}{1-e\cos(\theta)}$ is the graph of a conic section with directix $x = -d$
- the graph $r = \frac{ed}{1+e\cos(\theta)}$ is the graph of a conic section with directix $x = d$
- the graph $r = \frac{ed}{1-e\sin(\theta)}$ is the graph of a conic section with directix $y = -d$
- the graph $r = \frac{ed}{1+e\sin(\theta)}$ is the graph of a conic section with directix $y = d$

In each case above, $(0, 0)$ is a focus of the conic and the number e is the eccentricity of the conic.

- If $0 < e < 1$, the graph is an ellipse whose major axis has length $\frac{2ed}{1-e^2}$ and whose minor axis has length $\frac{2ed}{\sqrt{1-e^2}}$
- If $e = 1$, the graph is a parabola whose focal diameter is $2d$.
- If $e > 1$, the graph is an ellipse whose major axis has length $\frac{2ed}{e^2-1}$ and whose minor axis has length $\frac{2ed}{\sqrt{e^2-1}}$

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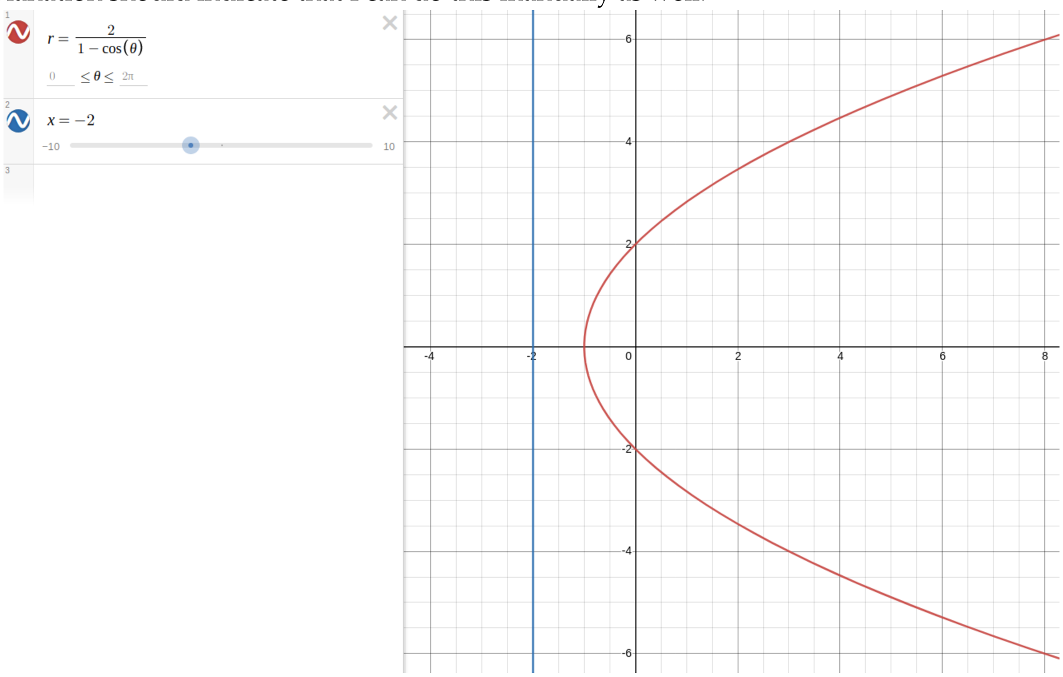
9. $r = \frac{2}{1-\cos(\theta)}$

These problems are thankfully a lot more simple. Upon inspection we can see a few things. Firstly, we can tell that $e = 1$ here which means we have a parabola. Nothing moves our focus, so that's at $(0,0)$. Since $e = 1$ that means $d = 2$ in this case. Due to that, our directrix is positioned at $x = -2$ making it a vertical line. Since our focus is to the right of the directrix our parabola will be opening to the right. Since the distance from the vertex to the focus is the same as the distance from the directrix to the vertex, our vertex must be positioned at $(-1,0)$, right smack dab in the middle of the two! Lastly we need the focal diameter.

The focal diameter is 4 times the distance from the vertex to the focus. $4 \cdot 1 = 4$. Our focal radius then is 2. Based upon that information we can plot this bad boy. I'll be using desmos for simplicity but this explanation should indicate that I can do this manually as well.



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10. $r = \frac{3}{2+\sin(\theta)}$

Let's rewrite this badboy first and foremost.

$$r = \frac{\frac{3}{2}}{1 + \frac{1}{2} \cdot \sin(\theta)}$$

$e < 0$ here, which means we have an ellipse. Based on our above theorem, the directrix will be at $y = d$, and with $d = 3$ being the case here our directrix is at $y = 3$. Now we can calculate our major and minor axes.

Major Axis:

$$\text{major} = \frac{2 \cdot e \cdot d}{1 - e^2}$$
$$\text{major} = \frac{2 \cdot \frac{1}{2} \cdot 3}{1 - \left(\frac{1}{2}\right)^2}$$
$$\text{major} = 4$$

Minor Axis:

$$\text{minor} = \frac{2 \cdot e \cdot d}{\sqrt{1 - e^2}}$$
$$\text{minor} = \frac{3}{\frac{\sqrt{3}}{2}}$$
$$\text{minor} \approx 3.46$$

Now to calculate our various vertices and co-vertices using $r(\theta)$.

$$r(0) = \frac{3}{2+0}$$
$$r(0) = \frac{3}{2}$$

This corresponds to horizontal location. So we get $\left(\frac{3}{2}, 0\right)$

$$r\left(\frac{\pi}{2}\right) = \frac{3}{2+0}$$
$$r(0) = \frac{3}{2}$$

This corresponds to horizontal location but negative. So we get $\left(-\frac{3}{2}, 0\right)$

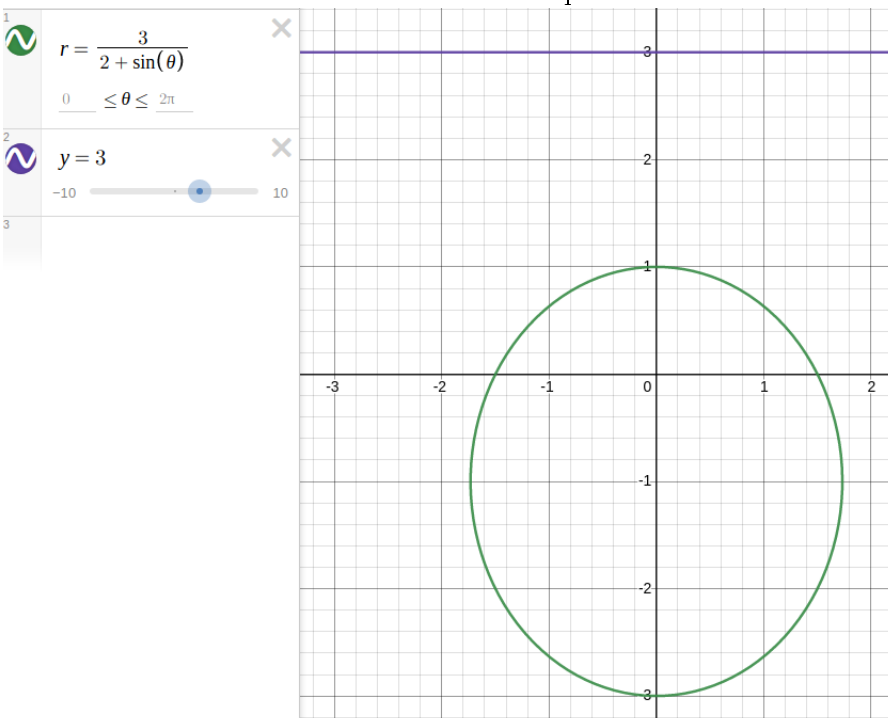
$$r(\pi) = \frac{3}{2+1}$$
$$r(\pi) = 1$$

This corresponds to vertical location. So we get $(0, 1)$

$$r\left(\frac{3\pi}{2}\right) = \frac{3}{2-1}$$
$$r\left(\frac{3\pi}{2}\right) = 3$$

This corresponds to vertical location but negative. So we get $(0, -3)$

Now we have all of the information we need to plot this one.



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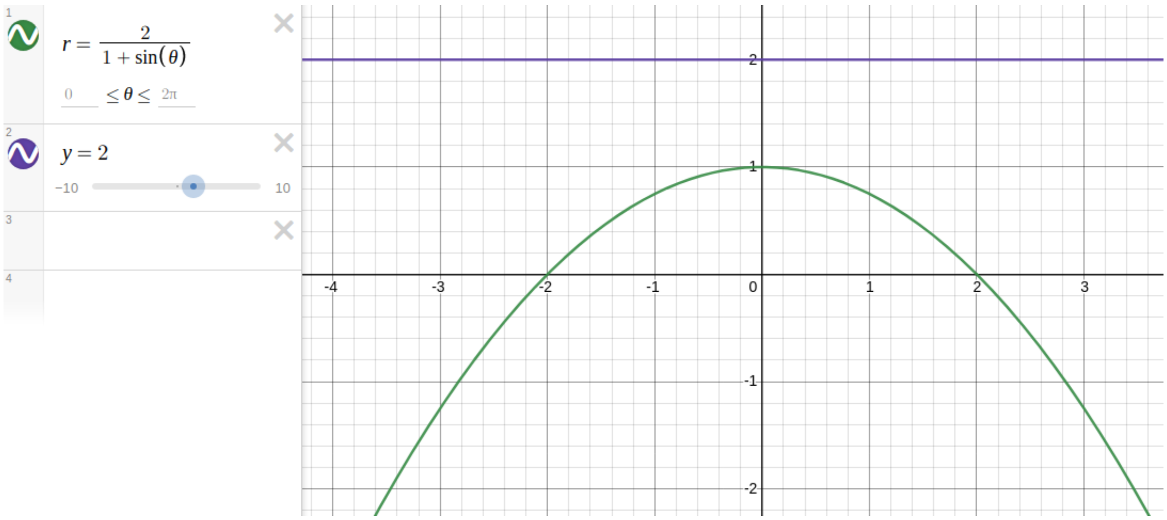
12. $r = \frac{2}{1+\sin(\theta)}$

Upon inspection we can pretty much solve this guy. $e = 1$ which means we have a parabola. As such $d = 2$ and our directrix is at $y = 2$ based on our given theorem. Our focus is at $(0, 0)$ which tells us our parabola is opening down AND the vertex must be at $(0, 1)$. Lastly our focal diameter is the same as in problem 9, so 4.

With this information I give the plot below.



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