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MATH 1120 SPRING 2021
Graphing Trig Functions Part 2
Feb 21st

Graphs of the Trigonometric Functions Part II:
10.5: 29–31, 35, 36, 38–40, 44–51

For exercises 29–31 show that the following functions are sinusoids by rewriting them in the forms $C(x) = A\cos(\omega x + \phi) + B$ and $S(x) = A\sin(\omega x + \phi) + B$ for $\omega > 0$ and $0 \le \phi < 2\pi$.

29 $f(x) = 2\sqrt{3}\cos(x) - 2\sin(x)$

Before we can start the actual process of solving this problem it would be wise to state information we will be utilizing.

Theorem 10.23. For $\omega > 0$, the functions

$$C(x) = A\cos(\omega x + \phi) + B \quad \text{and} \quad S(x) = A\sin(\omega x + \phi) + B$$

- have period $\frac{2\pi}{\omega}$
- have phase shift $-\frac{\phi}{\omega}$
- have amplitude $|A|$
- have vertical shift B

We're also going to need the sum formulas for both sine and cosine to solve this problem.

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$$
$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$

This problem allows us to make some assumptions so let's list off our givens. We can assume that B is 0 due to there being no vertical shift at the end of our function. I will also be assuming that $\omega = 1$ due to the structure of our function though this won't really be relevant due to how I will solve this.

$$B = 0 \qquad \omega = 1$$
$$A = ? \qquad \phi = ?$$

Okay, now we have the tools we need to work through this. The overall gameplan will be to take $C(x)$ and work it through the format of the expanded sum formulas. We will then substitute in values from $f(x)$. Doing this will allow us to solve for information we don't have, that being A and ϕ. We will first start with the expanded formula of cosine.

$$C(x) = A\cos(\omega x + \phi) + B$$

What's useful about $C(x)$ is that with a little work it fits very nicely into the format of our sum formulas listed above. For this we will let $\omega x = \alpha$ and $\phi = \beta$.

$$
\begin{aligned}
2\sqrt{3}\cos(x) - 2\sin(x) &= \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \\
2\sqrt{3}\cos(x) - 2\sin(x) &= A\cos(\omega x)\cos(\phi) + A\sin(\omega x)\sin(\phi) + B
\end{aligned}
$$

Since we're dealing with multiplication here we can rewrite this as:

$$
2\sqrt{3}\cos(x) - 2\sin(x) = A\cos(\phi)\cos(\omega x) + A\sin(\phi)\sin(\omega x) + B
$$

There's something else interesting here. We can equate the coefficients on either side of our equation. Doing so we can say the following:

$$
\begin{aligned}
2\sqrt{3}cos(x) &= A\cos(\phi)\cos(\omega x) \\
2\sqrt{3} &= A\cos(\phi) \\
2\sin(x) &= A\sin(\phi)\sin(\omega x) \\
2\sin(x) &= A\sin(\phi)
\end{aligned}
$$

We can actually use another identity now: the Pythagorean identity.

$$
\cos^2(\phi) + \sin^2(\phi) = 1
$$

Now, we can multiply this identity by A^2 and then substitute in our values.

$$
\begin{aligned}
A^2\cos^2(\phi) + A^2\sin^2(\phi) &= A^2 \\
(2\sqrt{3})^2 + 2^2 &= A^2 \\
12 + 4 &= A^2 \\
\pm 4 &= A
\end{aligned}
$$

Okay, that was a lot but we're finally getting somewhere here. We solved for A, now we can solve for ϕ. The thing to keep in mind is that we have two values of A here, a positive and a negative value. The value we choose ends up not mattering which I will showcase later. To start though, I will choose positive 4.

$$
\begin{aligned}
2\sqrt{3} &= A\cos(\phi) \\
2\sqrt{3} &= 4\cos(\phi) \\
\cos(\phi) &= \frac{2\sqrt{3}}{4} \\
\cos(\phi) &= \frac{\sqrt{3}}{2}
\end{aligned}
$$

$$
\begin{aligned}
A\sin(\phi) &= 2\sin(x) \\
4\sin(\phi) &= 2\sin(x) \\
\sin(\phi) &= \frac{1}{2}
\end{aligned}
$$

This step is easy, we just need to find the values on the unit circle where each of these is true.

$$
\begin{aligned}
\cos(\phi) &= \frac{\sqrt{3}}{2} \text{ when } \phi = \frac{\pi}{6} \text{ or } \frac{11\pi}{6} \\
\sin(\phi) &= \frac{1}{2} \text{ when } \phi = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}
\end{aligned}
$$

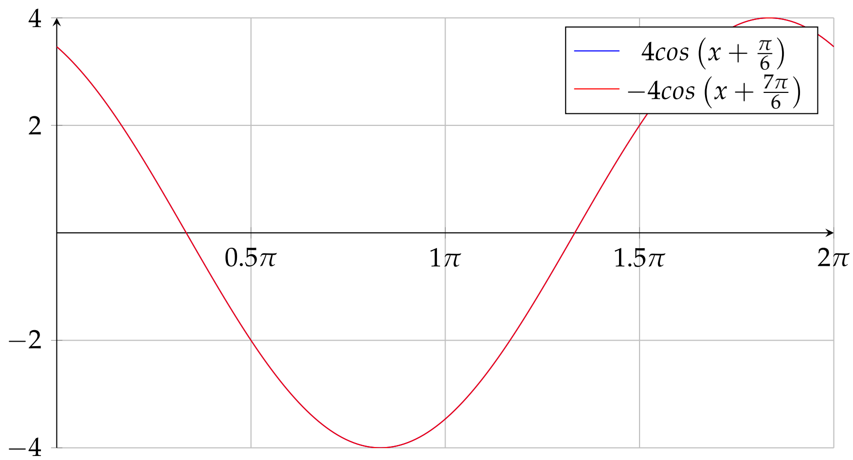
So, it seems like $\phi = \frac{\pi}{6}$ is our culprit. Perfect. We're not done yet though!

To be thorough let's go ahead and check $A = -4$ as well. I'll spare you the calculations, all we need to do is flip the signs. We get $\cos(\phi) = -\frac{\sqrt{3}}{2}$ and $\sin(\phi) = -\frac{1}{2}$.

$$\cos(\phi) = -\frac{\sqrt{3}}{2} \text{ when } \phi = \frac{5\pi}{6} \text{ or } \frac{7\pi}{6}$$
$$\sin(\phi) = -\frac{1}{2} \text{ when } \phi = \frac{11\pi}{6} \text{ or } \frac{7\pi}{6}$$

Our culprit now is $\phi = \frac{7\pi}{6}$. What's interesting here is that these are different values of ϕ. Let's plot them and see what's going on. Before we can do that though, let's finish the first part of this problem and plug these values back into $C(x)$.

$$C(x) = 4cos\left(x + \frac{\pi}{6}\right)$$
$$C(x) = -4cos\left(x + \frac{7\pi}{6}\right)$$



What we see here is that the two graphs overlap. They're equivalent functions. To describe what's going on the difference in phase shifts is, in a way, negated by the flip. As such both answers here are correct. Both ways of writing out $C(x)$ are equally valid!

So, I mentioned earlier that I simply assumed $\omega = 1$ but that how I solved the problem made that irrelevant. Let's go through and solve for ω just to be positive. To do this we will simply substitute back in everything we know into our formula and solve!

For the following, let $\alpha = \omega x$ and $\beta = \phi$ as always.

$$A\cos(\omega x + \phi) = 2\sqrt{3}\cos(x) - 2sin(x)$$
$$A\cos(\omega x)\cos(\phi) \mp A\sin(\omega x)\sin(\phi) = 2\sqrt{3}\cos(x) - 2sin(x)$$
$$A\cos(\phi)\cos(\omega x) \mp A\sin(\phi)\sin(\omega x) = 2\sqrt{3}\cos(x) - 2sin(x)$$
$$4\cos(\frac{11\pi}{6})cos(\omega x) \mp 4\sin(\frac{11\pi}{6})\sin(\omega x) = 2\sqrt{3}\cos(x) - 2sin(x)$$
$$4\left(\frac{\sqrt{3}}{2}\right)cos(\omega x) \mp 4\left(-\frac{1}{2}\right)\sin(\omega x) = 2\sqrt{3}\cos(x) - 2sin(x)$$
$$2\sqrt{3}cos(\omega x) \mp -2sin(\omega x) = 2\sqrt{3}\cos(x) - 2sin(x)$$

Note: Due to the $-2sin(x)$ on the righthand side we know that the $\mp$ we have on the lefthand side will be a plus sign. They have to be inverted.

$$2\sqrt{3}cos(\omega x) + -2sin(\omega x) = 2\sqrt{3}\cos(x) - 2sin(x)$$
$$2\sqrt{3}cos(\omega x) - 2sin(\omega x) = 2\sqrt{3}\cos(x) - 2sin(x)$$

The left and righthand side are now identical!

$$\therefore$$
$$\omega = 1$$

Okay, so we're on the fourth page and I need to reassure you that we're almost there. All we need to do now is go back to the beginning and work through $S(x)$ in the same way. It'll be faster this time though! So, of course first we need to equate $f(x)$ with $S(x)$. Also recall that $\omega = 1$ and $B = 0$.

$$f(x) = 2\sqrt{3}cos(x) - 2sin(x)$$
$$S(x) = A\sin(\omega x + \phi) + B$$
$$sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$$

We'll let $\alpha = \omega x$ amd $\beta = \phi$ just as last time.

$$2\sqrt{3}cos(x) - 2sin(x) = A\sin(\omega x - \phi) + B$$
$$2\sqrt{3}cos(x) - 2sin(x) = A\sin(\omega x)\cos(\phi) - A\cos(\omega x)sin(\phi) + B$$
$$2\sqrt{3}cos(x) - 2sin(x) = -A\cos(x)sin(\phi) + A\sin(x)\cos(\phi) + 0$$
$$2\sqrt{3}cos(x) - 2sin(x) = -Asin(\phi)\cos(x) + A\cos(\phi)\sin(x)$$

Alright, so now we equate the coefficients again. You know how it goes.

$$2\sqrt{3}cos(x) = -A\sin(\phi)\cos(x)$$
$$-2\sqrt{3} = A\sin(\phi)$$

$$2sin(x) = A\cos(\phi)\sin(x)$$
$$2 = A\cos(\phi)$$

Time to do the Pythagorean identity again just to be certain of our A value.

$$\cos^2(\phi) + \sin^2(\phi) = 1$$
$$A^2\cos^2(\phi) + A^2\sin^2(\phi) = A^2$$
$$(-2\sqrt{3})^2 + 2^2 = A^2$$
$$12 + 4 = A^2$$
$$A = \pm 4$$

Time to wrap this up.

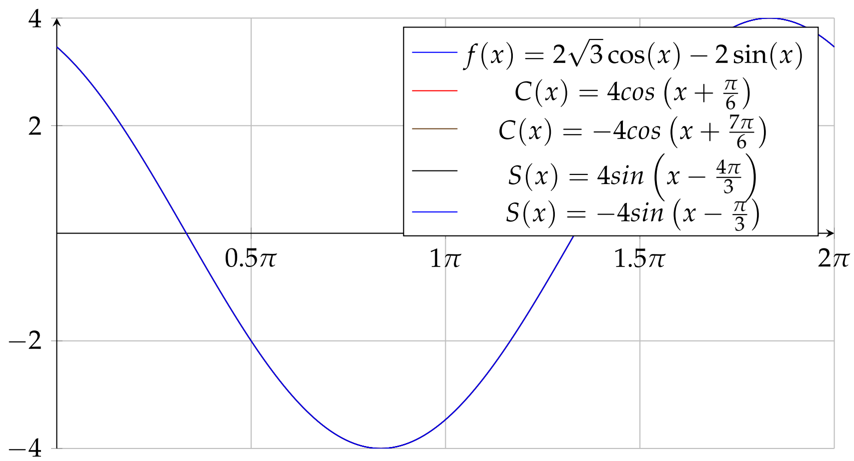
$$-2\sqrt{3} = 4\sin(\phi) \qquad -2 = 4\cos(\phi)$$
$$\sin(\phi) = -\frac{\sqrt{3}}{2} \qquad \cos(\phi) = -\frac{1}{2}$$
$$-2\sqrt{3} = -4\sin(\phi) \qquad -2 = -4\cos(\phi)$$
$$\sin(\phi) = \frac{\sqrt{3}}{2} \qquad \cos(\phi) = \frac{1}{2}$$

$$\sin(\phi) = -\frac{\sqrt{3}}{2} \text{ when } \phi = \frac{4\pi}{3} \text{ or } \frac{5\pi}{3}$$
$$\cos(\phi) = -\frac{1}{2} \text{ when } \phi = \frac{4\pi}{3} \text{ or } \frac{2\pi}{3}$$
$$\sin(\phi) = \frac{\sqrt{3}}{2} \text{ when } \phi = \frac{\pi}{3} \text{ or } \frac{2\pi}{3}$$
$$\cos(\phi) = \frac{1}{2} \text{ when } \phi = \frac{\pi}{3} \text{ or } \frac{5\pi}{3}$$

So, when $A = 4$ we know that $\phi = \frac{4\pi}{3}$ and when $A = -4$ $\phi = \frac{\pi}{3}$.
Plugging these values back into $S(x)$ we get the following:

$$S(x) = 4sin\left(x - \frac{4\pi}{3}\right)$$
$$S(x) = -4sin\left(x - \frac{\pi}{3}\right)$$

All five functions perfectly overlapping proves that we do in fact have five equivalent functions.

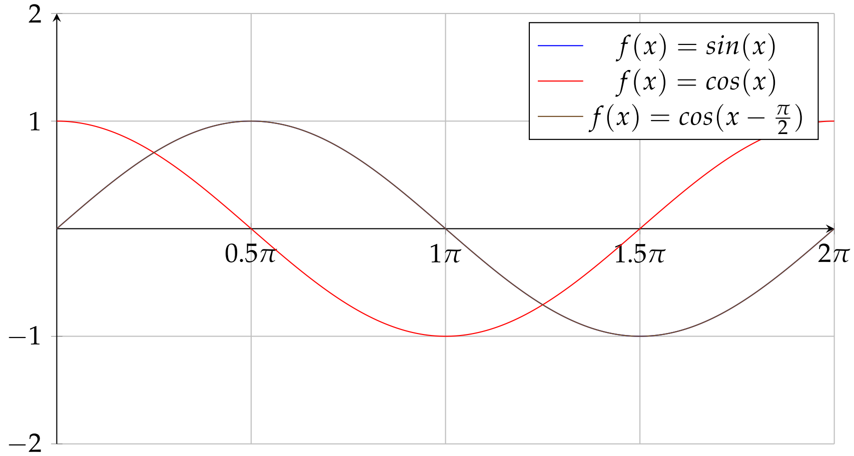


$$C(x) = 4cos\left(x + \frac{\pi}{6}\right)$$
$$C(x) = -4cos\left(x + \frac{7\pi}{6}\right)$$
$$S(x) = 4sin\left(x - \frac{4\pi}{3}\right)$$
$$S(x) = -4sin\left(x - \frac{\pi}{3}\right)$$

■

35 Show that if $f(x) = Asin(\omega x + \alpha) + B$, then $f(x) = Acos(\omega x + \beta) + B$ where $\beta = \alpha - \frac{\pi}{2}$.

This becomes easy to demonstrate when we look at what α in these functions represents, which is the phase shift. Sine and cosine are identical waves, sine is just shifted to the right by $\frac{\pi}{2}$. So it would stand to reason that when β in this instance is just $\alpha - \frac{\pi}{2}$ that you would get two equivalent waves. We can demonstrate this with a plot.



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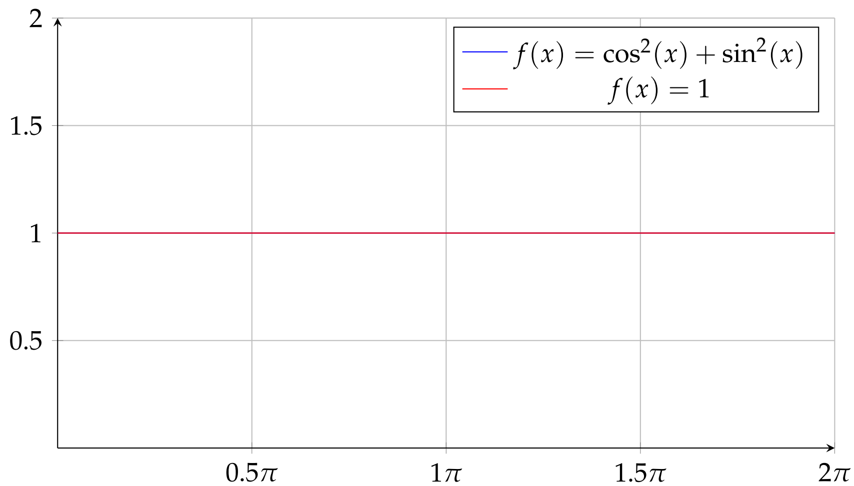
For exercises 38–40, verify the identity by graphing the right and left hand sides on a calculator.

38 $sin^2(\theta) + cos^2(\theta) = 1$



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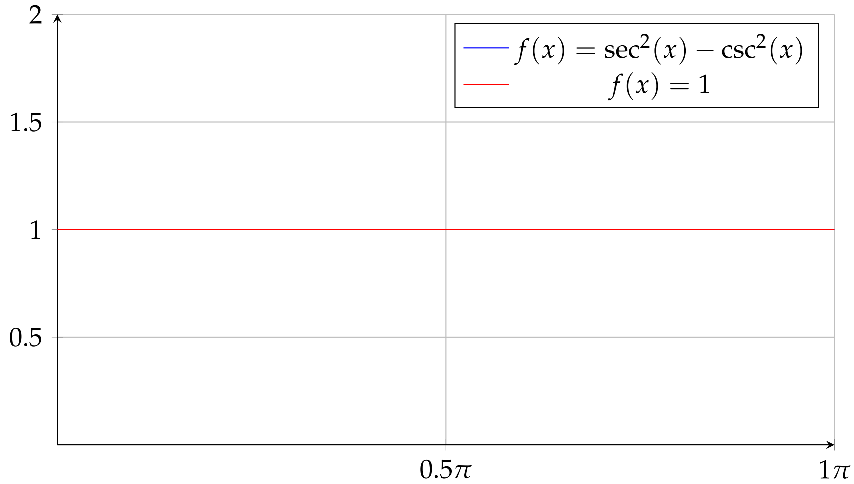
39 $sec^2(\theta) + tan^2(\theta) = 1$



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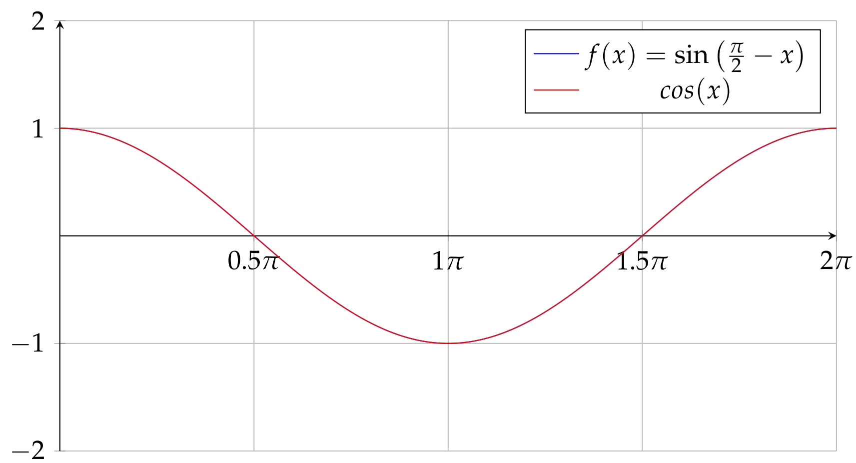
40 $\cos(x) = sin\left(\frac{\pi}{2} - 2\right)$



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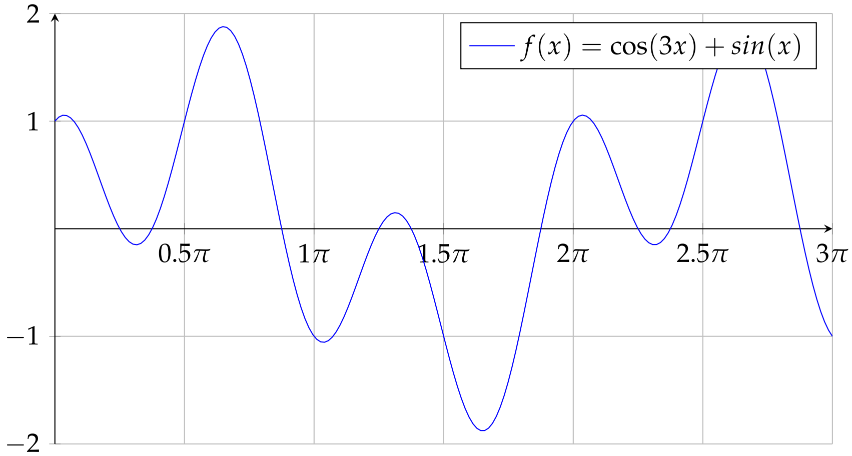
In exercises 44–50, graph the functions and discuss the given questions.

44 $f(x) = \cos(3x) + sin(x)$.
Is this function periodic? If so, what is the period?



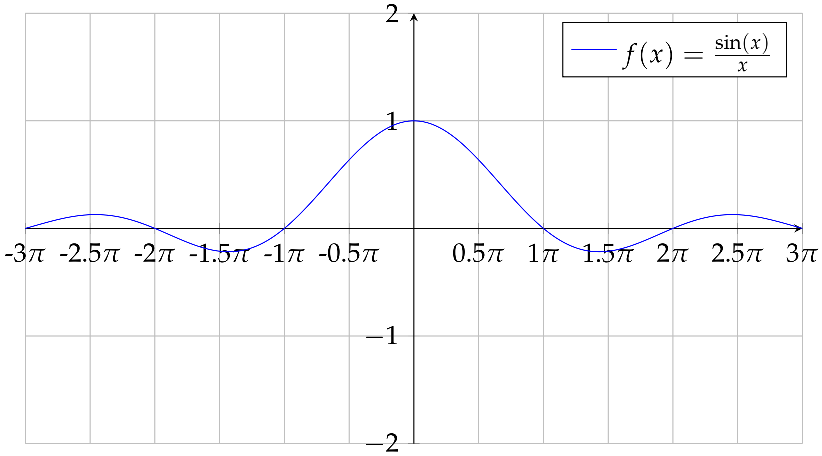
This function is periodic even if it doesn't appear to be at first glance. Going off of the plot alone the period seems to be right around 2π as normal. Looking at the function we know we're combining two periodic functions. $cos(3x)$ is going to repeat way faster than normal, whereas $sin(x)$ will have a standard period of 2π. So why does the overall function have a period of 2π? We can calculate it to find out. Let's get the period of our sine and cosine individually first.

$$3x_{cos} = 2\pi \qquad x_{sin} = 2\pi$$

$$x_{cos} = \frac{2\pi}{3}$$

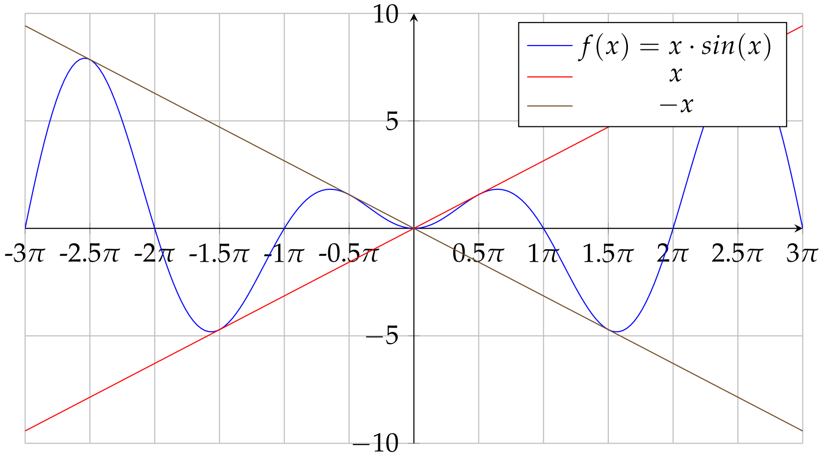
The period of $f(x)$ then is the least common multiple of x_{cos} and x_{sin}. Upon inspection we can see that the L.C.M is 2π. ∎

45 $f(x) = \frac{\sin(x)}{x}$
What appears to be the horizontal asymptote of the graph?



The horizontal asymptote appears to be at f(x) = 0. This makes sense as due to x being in the denominator it can never be zero, and as x grows the fraction will simply get closer and closer to zero as it reaches infinity in either direction. ∎

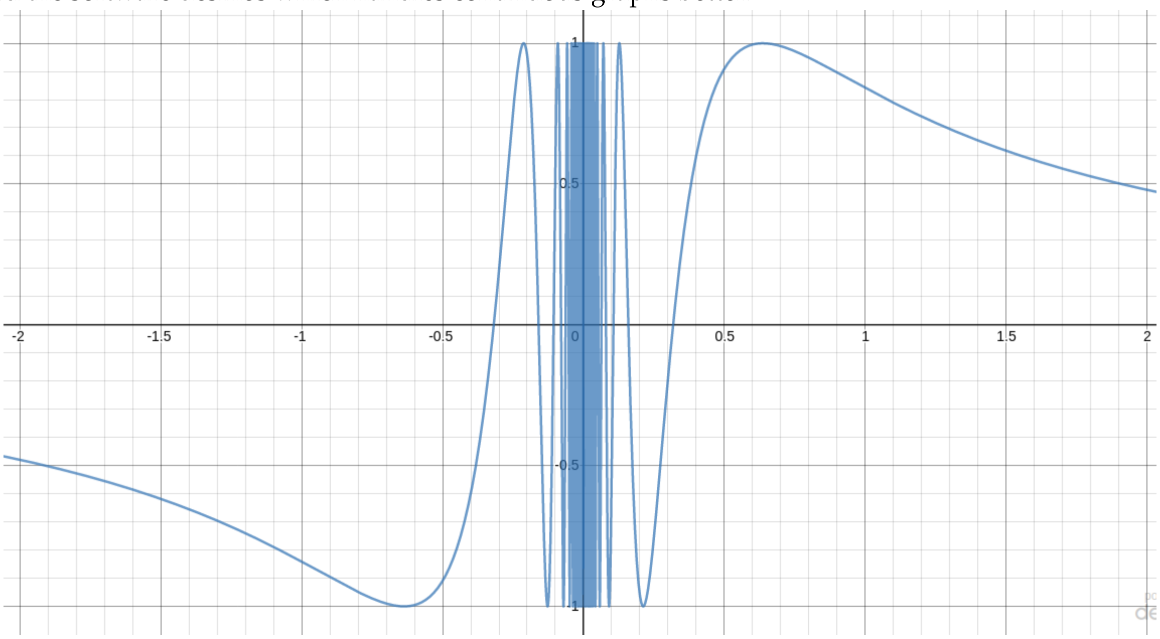
46 $f(x) = x \cdot sin(x)$.
Graph $y = \pm x$ on the same set of axes and describe the behavior of f.



So, what's happening here? So, first the obvious. As x grows the amplitude also grows. Also, the local maximums and minimums match up precisely with $y = \pm x$. This relationship will continue to hold true as x reaches infinity in either direction. $f(x)$ will, in a sense, never actually leave the boundaries set by $y = \pm x$. ∎

47 $f(x) = sin\left(\frac{1}{x}\right)$
What's happening as $x \to 0$?

What's happening here is that as x approaches zero the periodicity of the function gets bigger and bigger. This means it completes cycles faster and faster as we get closer to zero. I would describe the graph as violently oscillating between 1 and -1 at a speed that impossible for me to comprehend. The software I use for plots doesn't handle functions like this particularly well due to a limitation of given points. As such I used the software desmos which handles continuous graphs better.



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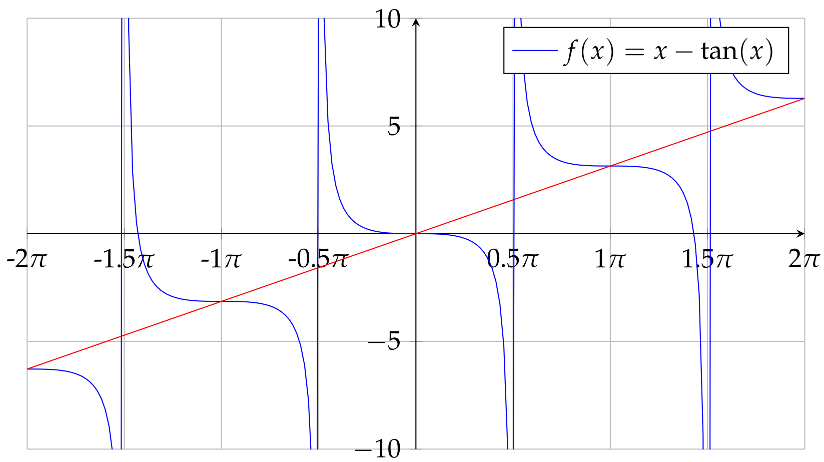
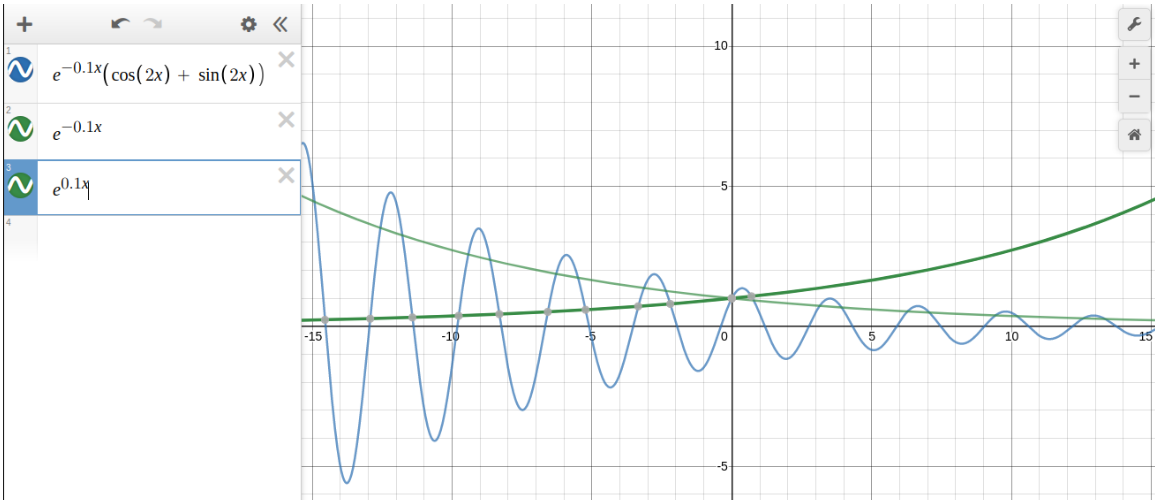
48 $f(x) = x - tan(x)$
Graph $y = x$ on the same set of axes and describe the behavior of f.



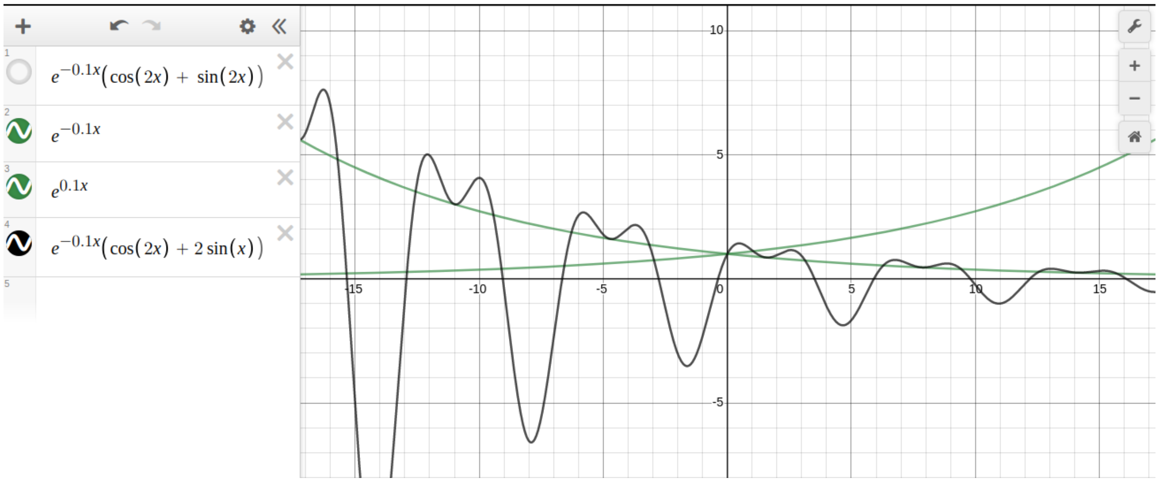
So, what's happening here is that the center point of each tangent period is increasing/decreasing alongside the $y = x$ function. Normally those center points are all right at $y = 0$. ■

49 $f(x) = e^{-0.1x}(\cos(2x) + sin(2x))$
Graph $y = \pm e^{-0.1x}$ and describe the behavior of f.



$f(x)$ intersects twice with $e^{-0.1x}$ near its local maximums consistently along the entire domain of $(-\infty, \infty)$. ∎

50 $f(x) = e^{-0.1x}(\cos(2x) + 2sin(x))$
Graph $y = \pm e^{-0.1x}$ and describe the behavior of f.



So, this may not be the best description but I would say that with every "period" of the graph you have bimodal humps that ride along $e^{-0.1x}$ as $f(x)$ goes from $-\infty \to 0$. This behavior continues along the entire domain but the amplitude goes down as x reaches ∞. ∎