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MATH 1120 SPRING 2021

Polar Coordinates and Graphs of Polar Equations
Section 11.4 Problems: 1–15 (Odd), 18–36 (Even), 37–44 (Odd), 77–83, 97, 98

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1 Problems 1–15

In Exercises 1–15, plot the point given in polar coordinates and then give three different expressions for the point such that:

a) $r < 0$ and $0 \leq \theta \leq 2\pi$

b) $r > 0$ and $\theta \leq 0$

c) $r > 0$ and $\theta \geq 2\pi$

Theorem 1 (Equivalent Representations of Points in Polar Coordinates)**.** Suppose (r, θ) and (r', θ') are polar coordinates where $r \neq 0$, $r' \neq 0$ and the angles are measured in radians. Then (r, θ) and (r', θ') determine the same point P if and only if one of the following is true.

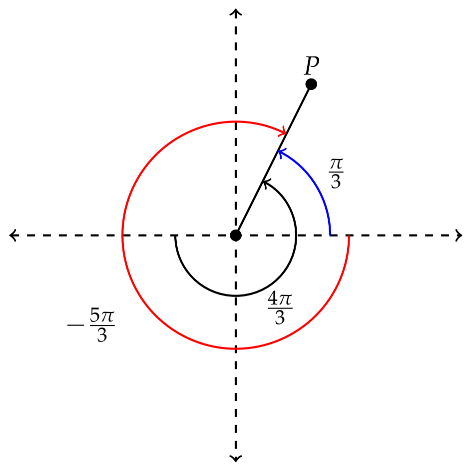
- $r' = r$ and $\theta' = \theta + 2\pi k$ for some integer k
- $r' = -r$ and $\theta' = \theta + (2k+1)\pi$ for some integer k

All polar coordinates of the form $(0, \theta)$ represent the pole regardless of the value of θ.

Before we get started let's breakdown exactly what we're doing here. With polar coordinates we're given a distance from the origin, "r", and a degree of rotation, "θ". Much like cartesian coordinates, the order in which we follow these directions is arbitrary. You can either travel the radius first and then account for the angle, or rotate first and then travel the radius.

This gets a little funny with a negative radius. You can either travel the radius, going left on the x axis, and then rotate. Or, you can rotate according to the angle and then just go backwards a distance of "r". Think of this like adjusting the angle of a car and then driving in reverse. It's roughly the same as turning your car around 180 degrees and driving forward. Both methods can get you to the same location. One's just a little (a lot) more dangerous and illegal than the other.

1. $P\left(2, \frac{\pi}{3}\right)$



a) $r < 0$ and $0 \leq \theta \leq 2\pi$

For this our radius will be negative so we can start our angle on the negative x axis and go counter-clockwise to reach our point. To get our new radius just flip our given radius' sign. To calculate θ we just add π to our given angle.

$$\frac{\pi}{3} + \frac{3\pi}{3}$$

$$\boxed{\left(-2, \frac{4\pi}{3}\right)}$$

b) $r > 0$ and $\theta \leq 0$

Here our radius will stay the same but we need a negative angle instead. To accomplish this we can just subtract 2π from our angle.

$$\frac{\pi}{3} - \frac{6\pi}{3}$$

$$\boxed{\left(2, -\frac{5\pi}{3}\right)}$$

c) $r > 0$ and $\theta \geq 2\pi$

This final part will have the same radius again, but we need an angle larger than 2π. We just now add 2π to our given angle.

$$\frac{\pi}{3} + \frac{6\pi}{3}$$

$$\boxed{\left(2, \frac{7\pi}{3}\right)}$$

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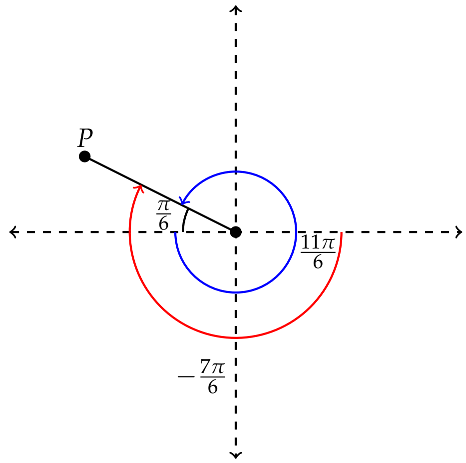
5. $P\left(12, -\frac{7\pi}{6}\right)$



a) $r < 0$ and $0 \leq \theta \leq 2\pi$

For this our radius will be negative so we can start our angle on the negative x axis and go counter-clockwise to reach our point. To get our new radius we can be a little clever. $-\frac{7\pi}{6}$ goes $\frac{\pi}{6}$ past the negative x-axis. Based on that we know that an angle wrapping all the way around to our point will be $2\pi - \frac{\pi}{6}$.

$$\frac{12\pi}{6} - \frac{\pi}{6}$$

$$\boxed{\left(-12, \frac{11\pi}{6}\right)}$$

b) $r > 0$ and $\theta \leq 0$

We can keep our radius the same and simply change our angle. To do so we can just subtract 2π from it to keep the angle equivalent.

$$-\frac{7\pi}{6} - \frac{12\pi}{6}$$

$$\boxed{\left(12, -\frac{19\pi}{6}\right)}$$

c) $r > 0$ and $\theta \geq 2\pi$

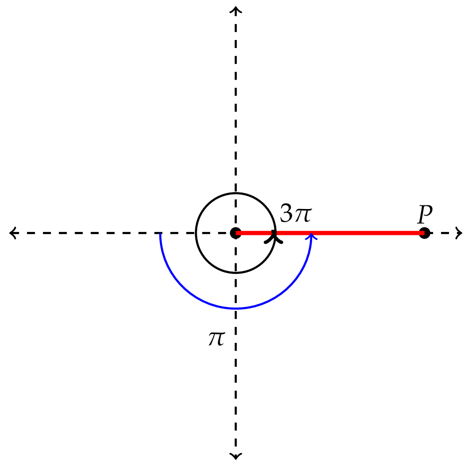
Same thing, different direction. We need to add 2π to our given angle. Sadly doing so will still result in an angle that's too small. So we need to do it twice.

$$-\frac{7\pi}{6} + 2 \cdot \left(\frac{12\pi}{6}\right)$$

$$\boxed{\left(12, \frac{17\pi}{6}\right)}$$

■

9. $P(-20, 3\pi)$



a) $r < 0$ and $0 \leq \theta \leq 2\pi$

Our radius is already negative so we just need a smaller, equivalent angle. By inspection we can tell that is simply 180° or 1π.

$$\boxed{(-20, \pi)}$$

b) $r > 0$ and $\theta \leq 0$

Since we have a positive radius now we start from the positive x-axis and just do one full loop around going clockwise for a negative angle. That's simply -2π. Keep it simple.

$$\boxed{(20, -2\pi)}$$

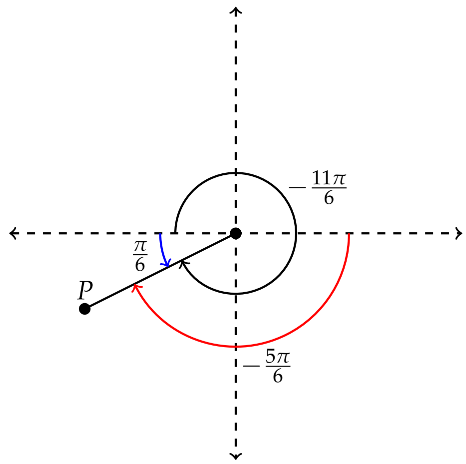
c) $r > 0$ and $\theta \geq 2\pi$

Same as last time except we just need that 2π to be positive.

$$\boxed{(20, 2\pi)}$$

■

13. $P\left(-3, -\frac{11\pi}{6}\right)$



a) $r < 0$ and $0 \leq \theta \leq 2\pi$

Our radius stays the same here. We now just need a positive angle. Since our given angle is $\frac{\pi}{6}$ short of one full rotation we know by inspection our new angle is exactly that, $\frac{\pi}{6}$.

$$\boxed{\left(-3, \frac{\pi}{6}\right)}$$

b) $r > 0$ and $\theta \leq 0$

Since we're starting from the positive x-axis now we can find our new angle by just taking π from our given angle. We skip over half a circles worth of rotation after all. Keep in mind this is a negative angle so we're actually adding.

$$-\frac{11\pi}{6} + \frac{6\pi}{6}$$

$$\boxed{\left(3, -\frac{5\pi}{6}\right)}$$

c) $r > 0$ and $\theta \geq 2\pi$

From the positive x-axis it takes us $\frac{7\pi}{6}$ to reach our point normally if we're using a positive angle. That's half a rotation plus the $\frac{\pi}{6}$ we covered in part a. From there we just need an equivalent angle that's greater than 2π. Easy.

$$\frac{7\pi}{6} + \left(\frac{12\pi}{6}\right)$$

$$\boxed{\left(3, \frac{19\pi}{6}\right)}$$

■

2 Problems 18–36

In Exercises 18–36, convert the point from polar coordinates into rectangular coordinates.

Theorem 2 (Conversion Between Rectangular and Polar Coordinates)**.** Suppose P is represented in rectangular coordinates as (x, y) and in polar coordinates (r, θ). Then:

- $x = r \cdot \cos(\theta)$ and $y = r \cdot \sin(\theta)$ $x^2 + y^2 = r^2$ and $\tan(\theta) = \frac{y}{x}$ (provided $x \neq 0$)

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18. $\left(2, \frac{\pi}{3}\right)$

Calculating x:

$$x = r \cdot \cos(\theta)$$
$$x = 2 \cdot \cos\left(\frac{\pi}{3}\right)$$
$$x = 1$$

Calculating y:

$$y = r \cdot \sin(\theta)$$
$$y = 2 \cdot \sin\left(\frac{\pi}{3}\right)$$
$$y = 2 \cdot \frac{\sqrt{3}}{2}$$
$$y = \sqrt{3}$$

$$\boxed{(1, \sqrt{3})}$$

■

22. $\left(-4, \frac{5\pi}{6}\right)$

Calculating x:

$i)\ x = r \cdot \cos(\theta)$

$ii)\ x = -4 \cdot \cos\left(\frac{5\pi}{6}\right)$

$iii)\ x = -4 \cdot -\frac{\sqrt{3}}{2}$

$iv)\ \boxed{x = 2\sqrt{3}}$

Calculating y:

$i)\ y = r \cdot \sin(\theta)$

$ii)\ y = -4 \cdot \sin\left(\frac{5\pi}{6}\right)$

$iii)\ y = -4 \cdot \frac{1}{2}$

$iv)\ \boxed{y = -2}$

$$\boxed{(2\sqrt{3}, -2)}$$

■

26. $(-117, -117\pi)$

Note: -117π radians is equivalent to $-\pi$ radians when calculating Cartesian coordinates!

Calculating x:

$$\begin{aligned}
x &= r \cdot \cos(\theta) \\
x &= -117 \cdot \cos(-117\pi) \\
x &= -117 \cdot -1 \\
x &= 117
\end{aligned}$$

Calculating y:

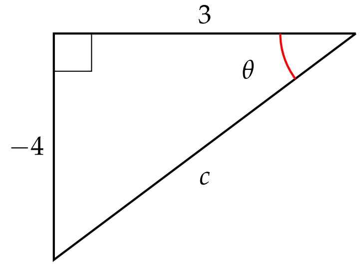
$$\begin{aligned}
y &= r \cdot \sin(\theta) \\
y &= -117 \cdot \sin(-\pi) \\
y &= -117 \cdot 0 \\
y &= 0
\end{aligned}$$

$$\boxed{(117, 0)}$$

■

30. $\left(5, \arctan\left(-\frac{4}{3}\right)\right)$

Note: Let $\arctan\left(-\frac{4}{3}\right) = \theta$



Calculating c:

$$c^2 = -4^2 + 3^2$$
$$c = 5$$

Calculating x:

$$x = r \cdot \cos(\theta)$$
$$x = 5 \cdot \cos\left(\arctan\left(-\frac{4}{3}\right)\right)$$
$$x = 5 \cdot \frac{3}{5}$$
$$x = 3$$

Calculating y:

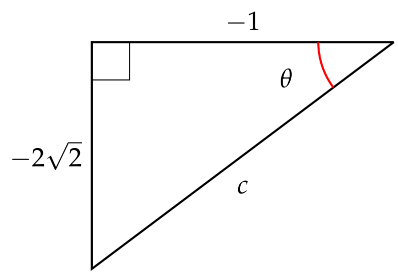
$$y = r \cdot \sin(\theta)$$
$$y = 5 \cdot \sin\left(\arctan\left(-\frac{4}{3}\right)\right)$$
$$y = 5 \cdot -\frac{4}{5}$$
$$y = -4$$

$$\boxed{(3, -4)}$$

■

34. $\left(\frac{2}{3}, \pi + \arctan\left(2\sqrt{2}\right)\right)$

Note: Let $\left(\pi + \arctan\left(2\sqrt{2}\right) = \theta\right)$. Due to the $+\pi$ we'll be moved from the first quadrant to the third, resulting in a negative cosine and sine.



Calculating c:

$$\begin{aligned} c^2 &= -1^2 + -\sqrt{2}^2 \\ c^2 &= 9 \\ c &= 3 \end{aligned}$$

Calculating x:

$$\begin{aligned} x &= r \cdot \cos(\theta) \\ x &= \frac{2}{3} \cdot \cos(\theta) \\ x &= \frac{2}{3} \cdot -\frac{1}{3} \\ x &= -\frac{2}{9} \end{aligned}$$

Calculating y:

$$\begin{aligned} y &= r \cdot \sin(\theta) \\ y &= \frac{2}{3} \cdot \sin(\theta) \\ y &= \frac{2}{3} \cdot -\frac{2\sqrt{2}}{3} \\ y &= -\frac{4\sqrt{2}}{9} \end{aligned}$$

$$\boxed{\left(-\frac{2}{9}, -\frac{4\sqrt{2}}{9}\right)}$$

■

3 Problems 37–44

In Exercises 37–44, convert the point from rectangular coordinates into polar coordinates with $r \geq 0$ and $0 \leq \theta \leq 2\pi$.

37 (0,5)

We can use the exact same tools to solve for different things here. It's fairly straightforward so let's get to it.

Calculating r:

$$\begin{aligned} r^2 &= x^2 + y^2 \\ r^2 &= 0^2 + 5^2 \\ r &= 5 \end{aligned}$$

Calculating θ part 1:

$$\begin{aligned} x &= r \cdot \cos(\theta) \\ \theta &= \arccos\left(\frac{x}{r}\right) \\ \theta &= \arccos\left(\frac{0}{5}\right) \\ \theta &= \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \end{aligned}$$

Calculating θ part 2:

$$\begin{aligned} y &= r \cdot \sin(\theta) \\ \theta &= \arcsin\left(\frac{y}{r}\right) \\ \theta &= \arcsin\left(\frac{5}{5}\right) \\ \theta &= \frac{\pi}{2} \end{aligned}$$

$$\boxed{\left(5, \frac{\pi}{2}\right)}$$

■

41 (-3,0)

We can use the exact same tools to solve for different things here. It's fairly straightforward so let's get to it.

Calculating r:

$$\begin{aligned} r^2 &= x^2 + y^2 \\ r^2 &= -3^2 + 0^2 \\ r &= 3 \end{aligned}$$

Calculating θ part 1:

$$\begin{aligned} x &= r \cdot \cos(\theta) \\ \theta &= \arccos\left(\frac{x}{r}\right) \\ \theta &= \arccos\left(\frac{-3}{3}\right) \\ \theta &= \pi \end{aligned}$$

Calculating θ part 2:

$$\begin{aligned} y &= r \cdot \sin(\theta) \\ \theta &= \arcsin\left(\frac{y}{r}\right) \\ \theta &= \arccos\left(\frac{0}{3}\right) \\ \theta &= 0, \pi, 2\pi \end{aligned}$$

$$\boxed{(3, \pi)}$$

■

4 Problems 77–83

In Exercises 77–83, convert the equation from polar coordinates into rectangular coordinates.

77. $r = 7$

$$r = 7$$
$$r^2 = 49$$
$$r^2 = x^2 + y^2$$
$$\boxed{49 = x^2 + y^2}$$

■

79. $r = \sqrt{2}$

$$r = \sqrt{2}$$
$$r^2 = 2$$
$$r^2 = x^2 + y^2$$
$$\boxed{2 = x^2 + y^2}$$

■

81. $\theta = \frac{2\pi}{3}$

$$\theta = \frac{2\pi}{3}$$
$$\tan(\theta) = \tan\left(\frac{2\pi}{3}\right)$$
$$\tan(\theta) = -\sqrt{3}$$
$$\tan(\theta) = \frac{y}{x}$$
$$-\sqrt{3} = \frac{y}{x}$$
$$\boxed{y = x \cdot -\sqrt{3}}$$
$$\boxed{x = -\frac{y}{\sqrt{3}}}$$

■

83. $\theta = \frac{3\pi}{2}$

This one is deceptively simple. When $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, etc...\ x = 0$. We can also figure out from this that $r = y$.

$$\boxed{x = 0\ ,\ y = r}$$

■

97 Convert the origin $(0,0)$ into polar coordinates in four different ways.

Doing this is deceptively easy. We can describe $(0,0)$ as having moved 0 away from the origin. That means have have a radius of 0. Our angle meanwhile can be literally anything. As long as we do not stray from the origin we can spin however we like.

$$\boxed{(0,\pi),(0,2\pi),\left(0,\frac{16\pi}{3}\right),(0,0))}$$

■

98. Use the Law of Cosines to develop a formula for the distance between two points in polar coordinates.

Theorem 3. Law of Cosines: Given a triangle with angle-side opposite pairs $(\alpha,a),(\beta,b),(\gamma,c)$, the following equations hold:

$$a^2 = b^2 + c^2 - 2 \cdot b \cdot c \cdot cos(\alpha) \qquad b^2 = a^2 + c^2 - 2 \cdot a \cdot c \cdot cos(\beta) \qquad c^2 = a^2 + b^2 - 2 \cdot a \cdot b \cdot cos(\gamma)$$

or, solving for the cosine in each equation, we have

$$\cos(\alpha) = \frac{b^2 + c^2 - a^2}{2bc} \qquad \cos(\beta) = \frac{a^2 + c^2 - b^2}{2ac} \qquad \cos(\gamma) = \frac{a^2 + b^2 - c^2}{2ab}$$

Here was the first solution I came up with. It is simply a conversion of the distance formula to polar coordinates. I personally don't think this is sufficient though.

Normal distance formula:

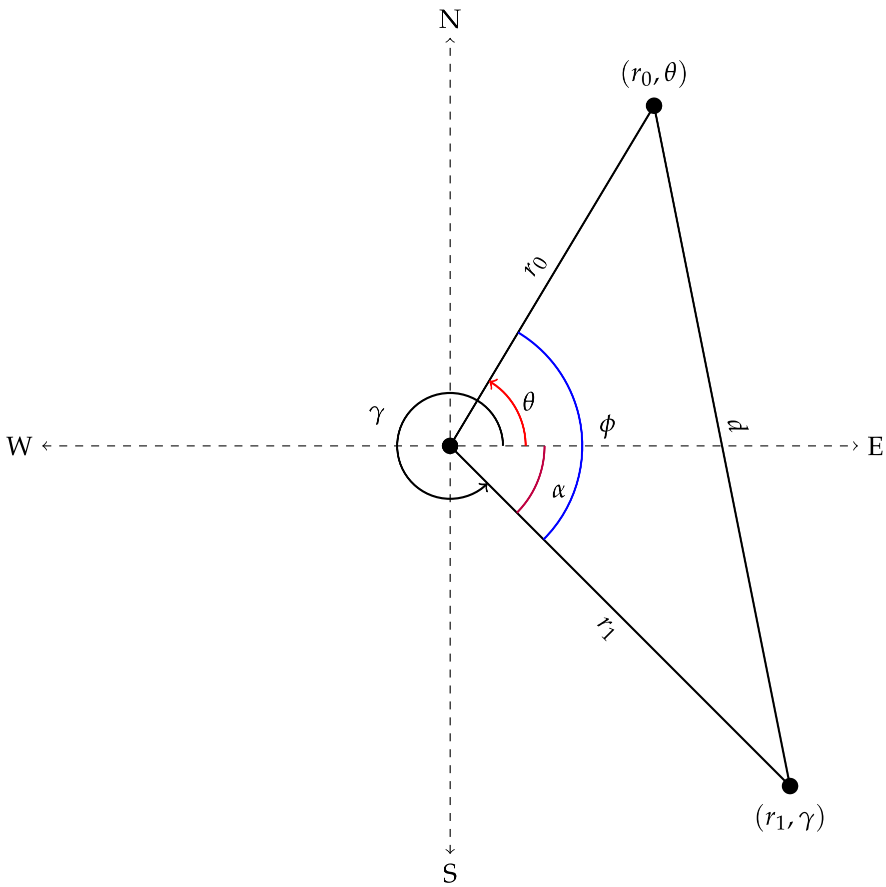
$$d_{cart} = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

To use this format we need to convert.

Converted to polar coordinates:

$$d_{polar} = \sqrt{((r_1\cos(\gamma)) - (r_0\cos(\theta)))^2 + ((r_1\sin(\gamma)) - (r_0\sin(\theta)))^2}$$

My second solution I think is more robust and ignores none of the given parameters in the problem. It would be best to represent this solution with a diagram. The breakdown will be on the following page.



Since both points we're using are based from the origin we can easily construct a triangle and solve using the law of cosines! The distance between points will be the side opposite our given angles. Since we have two sides represented by our two radii once we can get a working angle out of this mess we can solve. Let's do that.

Our first step is to solve for α using θ and γ! What we know about α is that, by construction, it is simply the difference between γ and a full rotation, otherwise represented by $2\pi - \gamma$ when $\gamma > 0$[1]. From there, we can solve for ϕ by simply adding α and θ together.

$$\alpha = 2\pi - \gamma$$
$$\phi = \alpha + \theta$$

From here we simply use the law of cosines.

$$d^2 = r_0^2 + r_1^2 - 2 \cdot r_0 \cdot r_1 \cdot \cos(\phi)$$
$$\boxed{d = \sqrt{r_0^2 + r_1^2 - 2 \cdot r_0 \cdot r_1 \cdot \cos(\phi)}}$$

∎

[1]When $\gamma < 0$ use $2\pi + \gamma$ instead