Brady Lamson
MATH 1120 SPRING 2021
Polar Form of Complex Numbers

1 Theorems

Theorem 1 (The Modulus and Argument of Complex Numbers). Let $z = a + bi$ be a complex number with $a = Re(z)$ and $b = Im(z)$. That means a representing the real axis and b representing the imaginary axis. Let (r, θ) be a polar representation of the point with rectangular coordinates (a, b) where $r \geq 0$.

- The **modulus** of z, denoted $|z|$, is defined by $|z| = r$.
- The angle θ is an **argument** of z. The set of all arguments of z is denoted $arg(z)$.
- If $z \neq 0$ and $-\pi < \theta \leq \pi$, then θ is the **principal argument** of z, written $\theta = Arg(z)$.

Theorem 2 (Properties of the Modulus). Let z and w be complex numbers.

- $|z|$ is the distance from z to 0 in the complex plain.
- $|z| \geq 0$ and $|z| = 0$ if and only if $z = 0$
- $|z| = \sqrt{Re(z)^2 + Im(z)^2}$
- **Product Rule:** $|zw| = |z||w|$
- **Power Rule:** $|z^n| = |z|^n$ for all natural numbers, n
- **Quotient Rule:** $|\frac{z}{w}| = \frac{|z|}{|w|}$, provided $w \neq 0$

Theorem 3 (Properties of the Argument). Let z be a complex number.

- If $Re(z) \neq 0$ and $\theta \in arg(z)$, then $\tan(\theta) = \frac{Im(z)}{Re(z)}$.
- If $Re(z) = 0$ and $Im(z) > 0$, then $arg(z) = \{\frac{\pi}{2} + 2\pi k | k \text{ is an integer } \}$.
- If $Re(z) = 0$ and $Im(z) < 0$, then $arg(z) = \{-\frac{\pi}{2} + 2\pi k | k \text{ is an integer } \}$.
- If $Re(z) = Im(z) = 0$, then $z = 0$ and $arg(z) = (-\infty, \infty)$.

Theorem 4 (A Polar Form of a Complex Number). Suppose z is a complex number and $\theta \in arg(z)$. The following expression is called a polar form for z:

$$|z|cis(\theta) = |z|[\cos(\theta) + i\sin(\theta)]$$

---

2 Exercises 1–20

In exercises 1–20, find a polar representation for the complex number z and then identify $Re(z)$, $Im(z)$, $|z|$, $arg(z)$ and $Arg(z)$.

1. $z = 9 + 9i$

The vast majority of this problem can be solved by utilizing the information provided in **theorem 1**. As such I'll be doing just that. I know the problem states to find a polar representation first, but I'll be doing it last. Let's first define our variables and figure out some basic stuff.

Our given format for a complex number is $z = a + bi$ and we also know that $a = Re(z)$ and $b = Im(z)$. Since this first problem is nice and clean we can easily state the following.

$$z = a + bi$$
$$a = 9 \text{ and } b = 9$$
$$\boxed{9 = Re(z)} \text{ and } \boxed{9 = Im(z)}$$

We can can also represent this equation in rectangular coordinates as $P(a, b)$ and, by extension, $P(9, 9)$. This is particularly useful to do as it gives us insight into what quadrant we're in. Since both a and b are positive, we're in quadrant 1. Next we'll solve for r.

$$r^2 = a^2 + b^2$$
$$r^2 = 9^2 + 9^2$$
$$r = \sqrt{162}$$
$$\boxed{r = \pm 9\sqrt{2}}$$

For **theorem 1** to properly work we need a positive value of r, so we'll choose that one.
Since $|z| = r$:

$$\boxed{|z| = 9\sqrt{2}}$$

Next up is θ.

$$\tan(\theta) = \frac{b}{a}$$
$$\tan(\theta) = \frac{9}{9}$$
$$\tan(\theta) = 1$$
$$\boxed{\theta = \frac{\pi}{4} + 2\pi k \text{ for integers k}}$$

$arg(z)$ uses a fairly similar format to the above, we just adjust it as such.

$$\boxed{arg(z) = \left\{ \frac{\pi}{4} + 2\pi k \mid \text{k is an integer} \right\}}$$

We're just about done here. Next up is the principal argument, $Arg(g)$, which limits θ to being between $-\pi$ and π. That one's easy. It's just $\frac{\pi}{4}$.

$$\boxed{Arg(z) = \frac{\pi}{4}}$$

Lastly, we finally have all the tools we need to fit this into a polar representation. We refer to **theorem 4** for the format we desire. Double checking with the answer in the back of the book helps as well! It's important to note that "cis" in this instance is an abbreviated form of $\cos(\theta) + i\sin(\theta)$ and that abbreviated form is what we need.

$$|z|cis(\theta) = |z|[\cos(\theta) + i\sin(\theta)]$$

$$\boxed{\text{Polar Rep. } = 9\sqrt{2}\, cis\left(\frac{\pi}{4}\right)}$$

■

2. $z = 5 + 5i\sqrt{3}$

Please refer to problem 1 for more detailed logic.

$$z = a + bi$$
$$a = 5 \text{ and } b = 5\sqrt{3}$$
$$\boxed{5 = Re(z)} \text{ and } \boxed{5\sqrt{3} = Im(z)}$$

Calculating r:

$$r^2 = a^2 + b^2$$
$$r^2 = 5^2 + (5\sqrt{3})^2$$
$$r = \sqrt{25 + 75}$$
$$r = \pm 10$$

$$\boxed{|z| = 10}$$

According to **theorem 3**: Since $Re(z) \neq 0$ that means that $\tan(\theta) = \frac{Im(z)}{Re(z)} = \frac{b}{a}$.
Calculating θ:

$$\tan(\theta) = \frac{b}{a}$$
$$\tan(\theta) = \frac{5\sqrt{3}}{5}$$
$$\tan(\theta) = \sqrt{3}$$

$$\boxed{\theta = \frac{\pi}{3} + 2\pi k \text{ for integers k}}$$

$$\boxed{arg(z) = \left\{\frac{\pi}{3} + 2\pi k \mid \text{k is an integer}\right\}}$$

$$\boxed{Arg(z) = \frac{\pi}{3}}$$

$$\boxed{\text{Polar Rep. } = 10\ cis\left(\frac{\pi}{3}\right)}$$

■

3. $z = 6i$

Please refer to problem 1 for more detailed logic.

$$z = a + bi$$
$$a = 0 \text{ and } b = 6$$
$$\boxed{0 = Re(z)} \text{ and } \boxed{6 = Im(z)}$$

Theorem 3: Since $Re(z) = 0$ and $Im(z) > 0$ then $\boxed{arg(z) = \left\{\frac{\pi}{2} + 2\pi k |\text{ k is an integer}\right\}}$

From here we can determine $Arg(z)$ as well, which is $\boxed{\frac{\pi}{2}}$.

Calculating r.

$$a^2 + b^2 = r^2$$
$$0^2 + 6^2 = r^2$$
$$\boxed{r = 6 = |z|}$$

Last is our polar representation.

$$\boxed{\text{Polar Rep. } = 6\ cis\left(\frac{\pi}{2}\right)}$$

■

4. $z = -3\sqrt{2} + 3i\sqrt{2}$

Please refer to problem 1 for more detailed logic.

$$z = a + bi$$
$$a = -3\sqrt{2} \text{ and } b = 3\sqrt{2}$$
$$\boxed{-3\sqrt{2} = Re(z)} \text{ and } \boxed{3\sqrt{2} = Im(z)}$$

Calculating r and $|z|$:

$$a^2 + b^2 = r^2$$
$$(-3\sqrt{2})^2 + (3\sqrt{2})^2 = r^2$$
$$\sqrt{36} = r$$
$$\boxed{\pm 6 = r}$$
$$\boxed{6 = |z|}$$

Calculating θ:

$$\tan(\theta) = \frac{Im(z)}{Re(z)} = \frac{b}{a}$$
$$\tan(\theta) = -\frac{3\sqrt{2}}{3\sqrt{2}}$$
$$\tan(\theta) = -1$$
$$\theta = \frac{3\pi}{4} + \pi k \text{ for integers k}$$

$$\boxed{arg(z) = \left\{ \frac{3\pi}{4} + \pi k | \text{ k is an integer} \right\}}$$

$\frac{3\pi}{4}$ is between $-\pi$ and π so we can say:

$$\boxed{Arg(z) = \frac{3\pi}{4}}$$

$$\boxed{\text{Polar Rep. } = 6\ cis \left(\frac{3\pi}{4} \right)}$$

∎

3 Exercises 21–40

In Exercises 21–40, find the rectangular form of the given complex number. Use whatever identities are necessary to find the exact values.

21. $z = 6cis(0)$

So, based on the format of the polar representation we know a few things. The first number, 6, is our radius and the second number, 0, is our θ. With this knowledge we can start to work backwards.

The big thing we need to do is rewrite "cis" as "cos" and "sin".

$$\begin{aligned} z = 6cis(0) &= |z|(\cos(\theta) + i\sin(\theta)) \\ &= 6 \cdot (\cos(0) + i\sin(0)) \\ &= 6 + i \cdot 0 \end{aligned}$$

$$\boxed{Re(z) = 6} \text{ and } \boxed{Im(z) = 0}$$

From this we can rewrite our equation in rectangular form.

$$\boxed{z = 6}$$

■

22. $z = 2cis\left(\frac{\pi}{6}\right)$

Same thing as the previous problem except we have a non-zero angle to work with. Based on our given information we know that:

$$\boxed{r = 2} \text{ and } \boxed{\theta = \frac{\pi}{6}}$$

$r = |z|$ so we can rewrite our equation and finish off the problem.

$$\begin{aligned} z = 2cis\left(\frac{\pi}{6}\right) &= |z|(\cos(\theta) + i\sin(\theta)) \\ z &= 2 \cdot \left(\cos\left(\frac{\pi}{6}\right) + i \cdot \sin\left(\frac{\pi}{6}\right)\right) \\ z &= 2 \cdot \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) \end{aligned}$$

$$\boxed{z = \sqrt{3} + i}$$

■

4 Exercises 41–52

In exercises 41–52, use the below given values of z and w to compute the quantity. Express your answer in polar form using the principal argument.

$$\boxed{z = -\frac{3\sqrt{3}}{2} + \frac{3}{2}i}$$

$$\boxed{w = 3\sqrt{2} - 3i\sqrt{2}}$$

41. zw

To start it is advisable to convert both z and w to polar form to make some calculations easier. This requires some legwork, but nothing we haven't done before. If you need clarification on my process for this please refer back to **problem 1**. First let's tackle z.

$$z = -\frac{3\sqrt{3}}{2} + \frac{3}{2}i$$
$$z = a + bi$$
$$\boxed{-\frac{3\sqrt{3}}{2} = Re(z)} \text{ and } \boxed{\frac{3}{2} = Im(z)}$$

Calculating $|z|$:

$$|z|^2 = Re(z)^2 + Im(z)^2$$
$$|z|^2 = \left(-\frac{3\sqrt{3}}{2}\right)^2 + \left(\frac{3}{2}\right)^2$$
$$|z| = \sqrt{\frac{27}{4} + \frac{9}{4}} = \sqrt{\frac{36}{4}} = \sqrt{9} = 3$$
$$\boxed{|z| = 3}$$

Calculating θ:

$$\tan(\theta) = \frac{Im(z)}{Re(z)}$$
$$\tan(\theta) = -\frac{\frac{3}{2}}{\frac{3\sqrt{3}}{2}}$$
$$= -\frac{3}{2} \cdot \frac{2}{3\sqrt{3}}$$
$$= -\frac{\sqrt{3}}{3}$$
$$\boxed{\theta = \frac{5\pi}{6} + \pi k \text{ for integers k}}$$

$$\boxed{\text{Polar Rep of z.} = 3\, cis\left(\frac{5\pi}{6}\right)}$$

Now we do the same thing for w.

$$w = 3\sqrt{2} - 3i\sqrt{2}$$

$$w = a + bi$$

$$\boxed{3\sqrt{2} = Re(w)} \text{ and } \boxed{-3\sqrt{2} = Im(w)}$$

Calculating $|w|$:

$$|w|^2 = Re(w)^2 + Im(w)^2$$
$$|w|^2 = (3\sqrt{2})^2 + (-3\sqrt{2})^2$$
$$|w| = \sqrt{36}$$

$$\boxed{|w| = 6 \text{ and } r = \pm 6}$$

Let ϕ be the angle of w in polar coordinates. For $\tan(\phi)$ we can be lazy and avoid calculations. Upon inspection it is clear that since both $Re(w)$ and $Im(w)$ are the same with one being negative. As such $tan(\phi) = -1$. Since $Re(w)$ is positive and $Im(w)$ is negative that places us in quadrant 3. Since ϕ needs to be between $-\pi$ and π we can state:

$$\boxed{\phi = -\frac{\pi}{4} + \pi k \text{ for integers k.}}$$

$$\boxed{\text{Polar Rep of } w \ = 6\ cis\left(-\frac{\pi}{4}\right)}$$

From here the rest is easy. We just multiply z and w together, simplify it down as much as possible and we get our solution.

$$z \cdot w = 3cis\left(\frac{5\pi}{6}\right) \cdot 6cis\left(-\frac{\pi}{4}\right)$$
$$= 18cis\left(\frac{5\pi}{6} - \frac{\pi}{4}\right) = 18cis\left(\frac{10\pi}{12} - \frac{3\pi}{12}\right)$$

$$\boxed{zw = 18cis\left(\frac{7\pi}{12}\right)}$$

■

81. Recall that given a complex number $z = a + bi$ it's complex conjugate, denoted $\overline{z}$, is given by $\overline{z} = a - bi$.

(a) Prove that $|\overline{z}| = |z|$

(b) Prove that $|z| = \sqrt{z \cdot \overline{z}}$

(c) Show that $Re(z) = \frac{z+\overline{z}}{2}$ and $Im(z) = \frac{z-\overline{z}}{2i}$

(d) Show that if $\theta \in arg(z)$ then $-\theta \in arg(\overline{z})$. Interpret this result geometrically.

(a) Prove that $|\overline{z}| = |z|$:

$$\begin{aligned} z &= a + bi & \overline{z} &= a - bi \\ Re(z) &= a & Re(\overline{z}) &= a \\ Im(z) &= b & Im(\overline{z}) &= -b \end{aligned}$$

$$|z| = \sqrt{Re(z)^2 + Im(z)^2} = \sqrt{a^2 + b^2}$$
$$|\overline{z}| = \sqrt{Re(\overline{z})^2 + Im(\overline{z})^2} = \sqrt{a^2 + (-b)^2}$$
$$\sqrt{a^2 + (b)^2} = \sqrt{a^2 + (-b)^2}$$
$$|z| = |\overline{z}|$$

■

(b) Prove that $|z| = \sqrt{z \cdot \overline{z}}$:

$$\begin{aligned} \sqrt{z \cdot \overline{z}} &= \sqrt{(a + bi) \cdot (a - bi)} \\ &= \sqrt{a^2 - abi + abi - bi^2} \\ &= \sqrt{a^2 - bi^2} \\ i^2 &= -1 \\ \sqrt{z \cdot \overline{z}} &= \sqrt{a^2 + b^2} = |z| \end{aligned}$$

■

(c) Show that $Re(z) = \frac{z+\overline{z}}{2}$ and $Im(z) = \frac{z-\overline{z}}{2i}$:

$$\begin{aligned} Re(z) &= \frac{z + \overline{z}}{2} \\ &= \frac{(a + bi) + (a - bi)}{2} \\ &= \frac{2a}{2} = a \\ Re(z) &= a = \frac{z + \overline{z}}{2} \end{aligned}$$

$$\begin{aligned} Im(z) &= \frac{z-\overline{z}}{2i} \\ &= \frac{(a+bi)-(a-bi)}{2i} \\ &= \frac{2bi}{2i} = b \\ Im(z) &= b = \frac{z-\overline{z}}{2i} \end{aligned}$$

■

(d) Show that if $\theta \in arg(z)$ then $-\theta \in arg(\overline{z})$. Interpret this result geometrically.: .

For this we'll let ϕ be an angle for $\overline{z}$ and θ will be an angle for z.

$$Im(\overline{z}) = -Im(z)$$
$$Re(\overline{z}) = Re(z)$$

$$\theta = arctan\left(\frac{Im(z)}{Re(z)}\right)$$
$$\phi = arctan\left(\frac{Im(\overline{z})}{Re(\overline{z})}\right)$$

Due to the above relationships this can otherwise be written as:

$$\phi = arctan\left(\frac{-Im(z)}{Re(z)}\right)$$

Therefore:

$$\phi = -\theta$$

I'm not 100% sure where to go from here. How do I fit this into the argument format? I know this final step is incredibly simple but it just isn't quite clicking for me. ■